THE VECTOR k–CONSTRAINED KP HIERARCHY AND SATO'S GRASSMANNIAN

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Abstract. We use the representation theory of the infinite matrix group to show that (in the polynomial case) the n–vector k–constrained KP hierarchy has a natural geometrical interpretation on Sato's infinite Grassmannian. This description generalizes the the k–reduced KP or Gelfand–Dickey hierarchies.

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§1. Introduction.

It is well–known that the k–th Gelfand–Dickey hierarchy, which generalizes the Korteweg–de Vries (KdV) hierarchy, can be obtained as a reduction of the Kadomtsev–Petviashvili (KP) hierarchy. The latter is defined as the set of deformation equations

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L],$$

for the first order pseudo-differential operator

$$L \equiv L(t,\partial) = \partial + u_1(t)\partial^{-1} + u_2(t)\partial^{-2} + \cdots ,$$

here $\partial = \frac{\partial}{\partial t_1}$, $t = (t_1, t_2, \dots)$ and $(L^k)_+$ stands for the differential part of L^k. Now L dresses as $L = P\partial P^{-1}$ with

$$P \equiv P(t,\partial) = 1 + a_1(t)\partial^{-1} + a_2(t)\partial^{-2} + \cdots .$$

One can choose P in such a way that

$$P(t,z) = \frac{\tau(t - [z^{-1}])}{\tau(t)},$$

where $\tau(t) = \tau(t_1, t_2, t_3, \dots)$ is the famous τ–function, introduced by the Kyoto group [DJKM1-3] and $[z] = (z, \frac{z^2}{2}, \frac{z^3}{3}, \dots)$. Sato [S] showed that such a τ–function corresponds to a point of some infinite Grassmannian Gr (see e.g. [S,SW]). Let H be the space of formal Laurent series $\sum a_n t^n$, such that $a_n = 0$ for $n >> 0$. The points of Gr are those linear subspaces $W \subset H$ for which the projection π_+ of W into $H_+ = \{\sum a_n t^n \in H | a_n = 0 \text{ for all } n < 0\}$ is a Fredholm operator. The k–th reduction or k–th Gelfand–Dickey hierarchy is obtained by assuming that

$$L^k = (L^k)_+,$$

which corresponds to a τ–function for which

$$\frac{\partial \tau}{\partial t_k} = \lambda\tau \qquad \text{for some } \lambda \in \mathbb{C}.$$

In the polynomial case, i.e. τ is a polynomial, clearly $\lambda = 0$. The point in the Grassmannian that corresponds to such a reduced τ–function satisfies

$$t^k W \subset W.$$

In recent years a lot of attention has been drawn to a new kind of reduction of the KP hierarchy, viz. the so–called k–constrained KP hierarchies [AFGZ,C,CWZ,CZ,D,DS,OS] (and references therein). Here one assumes that

$$L^k = (L^k)_+ + q\partial^{-1}r, \tag{1.1}$$

$q = q(t), r = r(t)$ being functions. Under this condition the KP hierarchy is constrained to

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad \frac{\partial q}{\partial t_k} = (L^k)_+ q \qquad \frac{\partial r}{\partial t_k} = -(L^k)_+^* r. \tag{1.2}$$

Here A^* stands for the adjoined operator of A (see e.g. [KV] for more details about pseudo–differential operators). The AKNS, Yajima–Oikawa and Melnikov hierarchies are some of the examples that appear amongst these constrained KP families.

In this paper we consider the generalization of this k–constrained KP hierarchy, wich was introduced by Sidorenko and Strampp in [SS], the n–vector k–constrained hierarchy. We assume that

$$L^k = (L^k)_+ + \sum_{j=1}^{n} q_j\partial^{-1}r_j, \tag{1.3}$$

then one obtains the following integrable system:

$$\frac{\partial L}{\partial t_k} = [(L^k)_+, L], \qquad \frac{\partial q_j}{\partial t_k} = (L^k)_+ q_j, \qquad \frac{\partial r_j}{\partial t_k} = -(L^k)_+^* r_j \qquad \text{for } 1 \le j \le n. \tag{1.4}$$

For $k = 1$ this hierarchy contains the coupled vector non–linear Schrödinger. Zhang and Cheng showed in [ZC] that if one asumes that

$$q_j(t) = \frac{\rho_j(t)}{\tau(t)} \qquad \text{and} \qquad r_j(t) = \frac{\sigma_j(t)}{\tau(t)}, \tag{1.5}$$

then L, q_j and r_j, $1 \le j \le n$ satisfy the n–vector k–constrained hierarchy if and only if $\tau(t)$, $\rho_j(t)$ and $\sigma_j(t)$ satisfy the following set of equations:

$$Res_{z=0} e^{-\eta(t,z)}\tau(t)e^{\xi(t,z)}e^{\eta(t',z)}\tau(t')e^{-\xi(t',z)} = 0, \tag{1.6}$$

$$Res_{z=0} z^k e^{-\eta(t,z)}\tau(t)e^{\xi(t,z)}e^{\eta(t',z)}\tau(t')e^{-\xi(t',z)} = \sum_{j=1}^{n}\rho_j(t)\sigma_j(t'), \tag{1.7}$$

$$Res_{z=0} z^{-1} e^{-\eta(t,z)}\tau(t)e^{\xi(t,z)}e^{\eta(t',z)}\rho_j(t')e^{-\xi(t',z)} = \rho_j(t)\tau(t'), \tag{1.8}$$

$$Res_{z=0} z^{-1} e^{-\eta(t,z)}\sigma_j(t)e^{\xi(t,z)}e^{\eta(t',z)}\tau(t')e^{-\xi(t',z)} = \tau(t)\sigma_j(t'). \tag{1.9}$$

where

$$\eta(t,z)=\sum_{i=1}^{\infty}\frac{\partial}{\partial t_i}\frac{z^{-i}}{i},\qquad \xi(t,z)=\sum_{i=1}^{\infty}t_iz^i \tag{1.10}$$

and $Res_{z=0}\sum_i a_iz^i=a_{-1}$.

In the case that $n=1$, Loris and Willox [LW] show that one can deduce some additional bilinear identities, but now involving $\frac{\partial\tau}{\partial t_k}$. It is unclear if this is possible for $n>1$, but we will not need these extra bilinear identities.

We will show in this paper that in fact L satisfies the n-vector k–constrained KP hierarchy (1.3-4) if and only if the corresponding point W in Gr has a linear subspace $W'\subset W$ of codimension n such that

$$t^kW'\subset W. \tag{1.10}$$

We will proof this only in the polynomial case, i.e. polynomial τ,ρ_j and σ_j, but we expect that this is still true in the non-polynomial case. We use the representation theory of the infinite–dimensional matrix group GL_∞, developed by Kac and Peterson [KP1-2] (see also [KR]), to achieve this result.

Notice that in this way we get a filtration of hierarchies, i.e., the n–vector k–constrained hierarchy is a subsystem of the $(n+1)$–vector k–constrained hierarchy, $n=0$ being the k–reduced KP or Gelfand–Dickey hierarchies.

§2. The semi-infinite wedge representation of the group GL_∞ and Sato's Grassmannian.

Consider the infinite complex matrix group

$$GL_\infty=\{A=(a_{ij})_{i,j\in\mathbb{Z}+\frac12}|A \text{ is invertible and all but a finite number of } a_{ij}-\delta_{ij} \text{ are } 0\}$$

and its Lie algebra

$$gl_\infty=\{a=(a_{ij})_{i,j\in\mathbb{Z}+\frac12}|\text{ all but a finite number of } a_{ij} \text{ are } 0\}$$

with bracket $[a,b]=ab-ba$. The Lie algebra gl_∞ has a basis consisting of matrices $E_{ij},\ i,j\in\mathbb{Z}+\frac12$, where E_{ij} is the matrix with a 1 on the (i,j)-th entry and zeros elsewhere. Let $\mathbb{C}^\infty=\bigoplus_{j\in\mathbb{Z}+\frac12}\mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis $\{v_j\}_{j\in\mathbb{Z}+\frac12}$. Both the group GL_∞ and its Lie algebra gl_∞ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k)=\delta_{jk}v_i.$$

The well-known semi–infinite wedge representation is constructed as follows [KP2] (see also [KR] and [KV]). The semi-infinite wedge space $F=\Lambda^{\frac12\infty}\mathbb{C}^\infty$ is the vector space

with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \dots$, where $i_1 > i_2 > i_3 > \dots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. We can now define representations R of GL_∞ and r of gl_∞ on F by

$$\begin{aligned} R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots , \\ r(a)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) &= \sum_k v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{k-1}} \wedge av_{i_k} \wedge v_{i_{k+1}} \wedge \cdots . \end{aligned} \tag{2.1}$$

These equations are related by the usual formula:

$$\exp(r(a)) = R(\exp a) \text{ for } a \in gl_\infty.$$

In order to perform calculations later on, it is convenient to introduce a larger group

$$\begin{aligned} \overline{GL_\infty} = \{A = (a_{ij})_{i,j\in\mathbb{Z}+\frac12} | &A \text{ is invertible and all but a finite} \\ &\text{number of } a_{ij} - \delta_{ij} \text{ with } i \ge j \text{ are } 0\} \end{aligned}$$

and its Lie algebra

$$\overline{gl_\infty} = \{a = (a_{ij})_{i,j\in\mathbb{Z}+\frac12} | \text{ all but a finite number of } a_{ij} \text{ with } i \ge j \text{ are } 0\}.$$

Both $\overline{GL_\infty}$ and $\overline{gl_\infty}$ act on a completion $\overline{\mathbb{C}^\infty}$ of the space $\mathbb{C}^\infty$, where

$$\overline{\mathbb{C}^\infty} = \{\sum_j c_j v_j | c_j = 0 \text{ for } j >> 0\}.$$

It is easy to see that the representations R and r extend to representations of $\overline{GL_\infty}$ and $\overline{gl_\infty}$ on the space F.

The representation r of gl_∞ and $\overline{gl_\infty}$ can be described in terms of wedging and contracting operators in F (see e.g. [KP2,KR]). Let v_j^* be the linear functional on $\mathbb{C}^\infty$ defined by $\langle v_i^*, v_j\rangle := v_i^*(v_j) = \delta_{ij}$ and let $\mathbb{C}^{\infty *} = \bigoplus_{j\in\mathbb{Z}+\frac12} \mathbb{C}v_j^*$ be the restricted dual of $\mathbb{C}^\infty$, then for any $w \in \mathbb{C}^\infty$, we define a wedging operator $\psi^+(w)$ on F by

$$\psi^+(w)(v_{i_1} \wedge v_{i_2} \wedge \cdots) = w \wedge v_{i_1} \wedge v_{i_2} \cdots . \tag{2.2}$$

Let $w^* \in \mathbb{C}^{\infty *}$, we define a contracting operator

$$\psi^-(w^*)(v_{i_1} \wedge v_{i_2} \wedge \cdots) = \sum_{s=1}^{\infty} (-1)^{s+1} \langle w^*, v_{i_s}\rangle v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots . \tag{2.3}$$

For simplicity we write

$$\psi_j^+ = \psi^+(v_{-j}), \qquad \psi_j^- = \psi^-(v_j^*) \qquad \text{for } j \in \mathbb{Z} + \frac12 \tag{2.4}$$

These operators satisfy the following relations ($i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -$):

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu}\delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$.

Introduce the following elements of F ($m \in \mathbb{Z}$):

$$|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots .$$

It is clear that F is an irreducible $\mathcal{C}\ell$-module generated by the vacuum $|0\rangle$ such that

$$\psi_j^\pm |0\rangle = 0 \text{ for } j > 0.$$

It is straightforward that the representation r is given by the following formula:

$$r(E_{ij}) = \psi_{-i}^+ \psi_j^- . \tag{2.5}$$

Define the *charge decomposition*

$$F = \bigoplus_{m\in\mathbb{Z}} F^{(m)} \tag{2.6}$$

by letting

$$\text{charge}(|0\rangle) = 0 \text{ and } \text{charge}(\psi_j^\pm) = \pm 1. \tag{2.7}$$

It is clear that the charge decomposition is invariant with respect to $r(g\ell_\infty)$ (and hence with respect to $R(GL_\infty)$). Moreover, it is easy to see that each $F^{(m)}$ is irreducible with respect to $g\ell_\infty$ (and GL_∞). Note that $|m\rangle$ is its highest weight vector, i.e.

$$\begin{aligned} &r(E_{ij})|m\rangle = 0 \text{ for } i < j, \\ &r(E_{ii})|m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m). \end{aligned}$$

Let $w \in F$, we define the Annihilator space $Ann(w)$ of w as follows:

$$Ann(w) = \{v \in \mathbb{C}^\infty | v \wedge w = 0\}. \tag{2.8}$$

Notice that $Ann(w) \neq 0$, since $v_j \in Ann(w)$ for $j << 0$. This Annihilator space for perfect (semi–infinite) wedges $w \in F^{(m)}$ is related to the GL_∞-orbit

$$\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}$$

of the highest weight vector $|m\rangle$ as follows. Let $A = (A_{ij})_{i,j\in\mathbb{Z}} \in GL_\infty$, denote by $A_j = \sum_{i\in\mathbb{Z}} A_{ij} v_i$ then by (2.8)

$$\tau_m = R(A)|m\rangle = A_{m-\frac{1}{2}} \wedge A_{m-\frac{3}{2}} \wedge A_{m-\frac{5}{2}} \wedge \cdots , \tag{2.9}$$

with $A_{-j} = v_{-j}$ for $j >> 0$. Notice that since τ_m is a perfect (semi-infinite) wedge

$$Ann(\tau_m) = \sum_{j<m} \mathbb{C}A_j \subset \mathbb{C}^\infty.$$

By identifying $v_i = t^{-i-\frac{1}{2}}$ for $i \in \mathbb{Z} + \frac{1}{2}$, we can write $A_j = A_j(t) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} A_{ij}t^{-i-\frac{1}{2}}$ as a Laurent polynomial in t. In this way we can identify $Ann(\tau_m)$ with a subspace $W_{\tau_m} = \sum_{j<m} \mathbb{C}A_j(t)$ of the space H of all Laurent polynomials. Notice that this space H differs from the one described in section 1. So from now on let Gr consist of all linear subspaces of H which contain

$$H_j := \sum_{i=-j}^{\infty} \mathbb{C}t^i$$

for $j >> 0$ and let $Gr = \cup_{m\in\mathbb{Z}} Gr_m$ (disjoint union) with

$$Gr_m = \{W \in Gr | H_j \subset W \text{ and } \dim W/H_j = m - j \text{ for } j << 0\},$$

then we can construct a cannonical map

$$\phi : \mathcal{O}_m \to Gr_m, \qquad \phi(\tau_m) = W_{\tau_m} := \sum_{i<m} \mathbb{C}A_i(t).$$

It is clear that $\phi(|m\rangle) = H_m$ and that ϕ is surjective with fibers $\mathbb{C}^\times$. This construction is due to Sato [S], we call Gr the polynomial Grassmannian. From now on we will call a perfect wedge also a τ–function (N.B. $\tau = 0$ is also a τ–function).

§3. The boson-fermion correspondence.

Introduce the fermionic fields ($z \in \mathbb{C}^\times$):

$$\psi^\pm[z] \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k^\pm z^{-k-\frac{1}{2}}. \tag{3.1}$$

Next we introduce bosonic fields:

$$\alpha[z] \equiv \sum_{k\in\mathbb{Z}} \alpha_k z^{-k-1} \stackrel{def}{=} :\psi^+[z]\psi^-[z]:, \tag{3.2}$$

where : : stands for the *normal ordered product* defined in the usual way ($\lambda, \mu = +$ or $-$):

$$:\psi_k^\lambda \psi_\ell^\mu: := \begin{cases} \psi_k^\lambda \psi_\ell^\mu & \text{if } \ell \geq k \\ -\psi_\ell^\mu \psi_k^\lambda & \text{if } \ell < k. \end{cases}$$

One checks (using e.g. Wick's formula) that the operators α_k satisfy the commutation relations of the associative oscillator algebra, one has:

$$[\alpha_k, \alpha_\ell] = k\delta_{k,-\ell} \quad \text{and} \quad \alpha_k|m\rangle = 0 \text{ if } k > 0. \tag{3.3}$$

In order to express the fermionic fields $\psi^\pm(z)$ in terms of the bosonic operators α_ℓ, we need some additional operator Q. This operators is uniquely defined as follows:

$$Q(v_{i_1} \wedge v_{i_2} \wedge \cdots) = (v_{i_1+1} \wedge v_{i_2+1} \wedge \cdots). \tag{3.4}$$

So

$$Q|0\rangle = |1\rangle, \ Q\psi_k^\pm = \psi_{k\mp 1}^\pm Q$$

and Q satisfies the following commutation relations with the α's:

$$[\alpha_k, Q] = \delta_{k0}Q.$$

In this paper the operator Q^{-k} will play an important role. If $w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge \cdots$ is a perfect wedge then

$$Q^{-k}(w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge \cdots) = \Lambda^k w_{m-\frac{1}{2}} \wedge \Lambda^k w_{m-\frac{3}{2}} \wedge \cdots , \tag{3.5}$$

where $\Lambda = \sum_{j\in\mathbb{Z}+\frac{1}{2}} E_{j,j+1}$.

Theorem 3.1. ([DJKM1], [JM])

$$\psi^\pm[z] = Q^{\pm 1} z^{\pm\alpha_0} \exp(\mp \sum_{k<0} \frac{1}{k}\alpha_k z^{-k}) \exp(\mp \sum_{k>0} \frac{1}{k}\alpha_k z^{-k}). \tag{3.6}$$

Proof. See [TV].

The operators on the right-hand side of (3.6) are called vertex operators. They made their first appearance in string theory (cf. [FK]).

We now describe the boson-fermion correspondence. Let $\mathbb{C}[t]$ be the space of polynomials in indeterminates $t = (t_1, t_2, t_3, \dots)$. Let $B = \mathbb{C}[q, q^{-1}, t] = \mathbb{C}[t] \otimes_\mathbb{C} \mathbb{C}[q, q^{-1}]$ be the tensor product of algebras. Then the boson-fermion correspondence is the vector space isomorphism

$$\sigma : F \xrightarrow{\sim} B,$$

given by

$$\sigma(\alpha_{-m_1} \dots \alpha_{-m_s}|k\rangle) = m_1 \dots m_s t_{m_1} \dots t_{m_s} q^k.$$

Notice that the power of q is the value of the charge. The transported action of the operators α_m and Q looks as follows:

$$\sigma Q \sigma^{-1} = q, \qquad \sigma \alpha_m \sigma^{-1} = \begin{cases} -mt_m & \text{if } m < 0, \\ \frac{\partial}{\partial t_m} & \text{if } m > 0, \\ q\frac{\partial}{\partial q} & \text{if } m = 0. \end{cases} \tag{3.7}$$

Hence

$$\sigma \psi^{\pm}[z] \sigma^{-1} = q^{\pm 1} z^{\pm q \frac{\partial}{\partial q}} e^{\pm \xi(t,z)} e^{\mp \eta(t,z)}, \tag{3.8}$$

with $\eta(t,z)$ and $\xi(t,z)$ given by (1.10)

§4. Identification of the bilinear identities.

From now on we assume that $\tau \in F^{(m)}$, hence that τ is the inverse immage under σ. Using the boson–fermion correspondence of the previous section, we rewrite the bilinear identities (1.6-9) of Zhang and Cheng now as equations in $F \otimes F$. Notice first the following equality of operators on $F \otimes F$:

$$Res_{z=0} \psi^+[z] \otimes \psi^-[z] = \sum_{i \in \mathbb{Z}+\frac{1}{2}} \psi_i^+ \otimes \psi_{-i}^-$$

Now (1.6-9) turn into the following equations:

$$\sum_{i \in \mathbb{Z}+\frac{1}{2}} \psi_i^+ \tau \otimes \psi_{-i}^- \tau = 0, \tag{4.1}$$

$$\sum_{i \in \mathbb{Z}+\frac{1}{2}} \psi_i^+ \tau \otimes \psi_{-i}^- Q^{-k} \tau = \sum_{j=1}^{n} \rho_j \otimes \sigma_j, \tag{4.2}$$

$$\sum_{i \in \mathbb{Z}+\frac{1}{2}} \psi_i^+ \tau \otimes \psi_{-i}^- \rho_j = \rho_j \otimes \tau, \tag{4.3}$$

$$\sum_{i \in \mathbb{Z}+\frac{1}{2}} \psi_i^+ \sigma_j \otimes \psi_{-i}^- Q^{-k} \tau = Q^{-k} \tau \otimes \sigma_j. \tag{4.4}$$

Here $Q^{-k}\tau \in F^{(m-k)}$, $\rho_j \in F^{(m+1)}$ and $\sigma_j \in F^{(m-k-1)}$ for all $1 \leq j \leq n$. Equation (4.1) is called the KP hierarchy in the fermionic picture, it characterizes the GL_∞-orbit $\mathcal{O}_m$, i.e.

Proposition 4.1. ([KP2]) *A non-zero element τ of $F^{(m)}$ lies in $\mathcal{O}_m$ if and only if τ satisfies the equation* (4.1).

If $\tau \in \mathcal{O}_m$, then we can write τ as a perfect wedge

$$\tau = w_{m-\frac{1}{2}} \wedge w_{m-\frac{3}{2}} \wedge w_{m-\frac{5}{2}} \wedge w_{m-\frac{7}{2}} \wedge \cdots, \tag{4.5}$$

such that $w_{-\ell} = v_{-\ell}$ for $\ell >> 0$. The corresponding point $W_\tau \in Gr_m$ is then given by

$$W_\tau = \langle w_{m-\frac{1}{2}}, w_{m-\frac{3}{2}}, w_{m-\frac{5}{2}}, w_{m-\frac{7}{2}}, \ldots \rangle. \tag{4.6}$$

The geometrical interpretation of (4.3-4) is given by the following proposition.

Proposition 4.2. *Let $\tau \in \mathcal{O}_m$, $\rho \in F^{(m+1)}$ and $\sigma \in F^{(m-1)}$, then*
(1) *τ and ρ satisfy*

$$\sum_{i\in\mathbb{Z}+\frac12} \psi_i^+\tau \otimes \psi_{-i}^-\rho = \rho\otimes\tau, \tag{4.7}$$

if and only if $\rho \in \mathcal{O}_{m+1}$ and $W_\tau \subset W_\rho$,
(2) *τ and σ satisfy*

$$\sum_{i\in\mathbb{Z}+\frac12} \psi_i^+\sigma \otimes \psi_{-i}^-\tau = \tau\otimes\sigma, \tag{4.8}$$

if and only if $\sigma \in \mathcal{O}_{m-1}$ and $W_\sigma \subset W_\tau$.

Proof. Without loss of generality we may assume (since the operator $\sum_i \psi_i^+ \otimes \psi_{-i}^-$ commutes with the action of $R(GL_\infty)\otimes R(GL_\infty))$ that $\tau = |m\rangle$. Then (4.7) is equivalent to

$$\sum_{i>m} v_i \wedge |m\rangle \otimes \psi_i^-\rho = \rho \otimes |m\rangle.$$

Since all elements $v_i\wedge|m\rangle$, for $i>m$, are linearly independent, we deduce that $\psi_i^-\rho = \lambda_i|m\rangle$ and that $\rho \in \langle v_i\wedge|m\rangle | i>m\rangle$. Hence $\rho = w\wedge|m\rangle$ for some $w\in\mathbb{C}^\infty$ and thus $\rho\in\mathcal{O}_{m+1}$ and $W_\tau\subset W_\rho$.

The converse, since $W_\tau \subset W_\rho$, $\rho = w\wedge|m\rangle$ for some $w \in \mathbb{C}^\infty$. Then

$$\begin{aligned}\sum_{i\in\mathbb{Z}+\frac12} \psi_i^+\tau\otimes\psi_{-i}^-(w\wedge\tau) &= (w\wedge\tau)\otimes\tau - (1\otimes\psi^+(w))(\sum_{i\in\mathbb{Z}+\frac12}\psi_i^+\tau\otimes\psi_{-i}^-\tau)\\ &= (w\wedge\tau)\otimes\tau\end{aligned}$$

For $\tau = |m\rangle$, (4.8) is equivalent to

$$\sum_{i<m} (v_i\wedge\sigma)\otimes\psi_i^-|m\rangle = |m\rangle\otimes\sigma.$$

Since the elements $\psi_i^-|m\rangle$ for $i<m$ are all linearly independent, we conclude that $v_i\wedge\sigma = \lambda_i|m>$ and that $\sigma\in\langle\psi_i^-|m\rangle|i<m\rangle$. Hence $\sigma = \sum_{i=-\infty}^{m-\frac12} a_i\psi_i^-|m\rangle$. Since $\sigma\in F^{(m-1)}$, $a_i = 0$ for all $i < -N << 0$. We now calculate $Ann(\sigma)$. Clearly $Ann(\sigma)\subset\langle v_i|i<m\rangle = Ann(|m\rangle)$, so let $v = \sum_{i<m}(-)^i b_i v_i$, then $\sum_{i=-N+\frac12}^{m-\frac12} a_i b_i = 0$. Hence, if $\sigma\neq 0$, we only find one restriction for the collection of b_i's, from which we conclude that σ is a perfect wedge. The converse of this statement follows immediately by writing $\tau = w\wedge\sigma$. □

We next proof the following

Proposition 4.3. *Let $\tau \in \mathcal{O}_m$, $\rho_j \in \mathcal{O}_{(m+1)}$ and $\sigma_j \in \mathcal{O}_{(m-k-1)}$, $1 \le j \le n$, be related by*

$$W_\tau \subset W_{\rho_j}, \qquad W_{\sigma_j} \subset \Lambda^k W_\tau, \tag{4.9}$$

then τ satisfies equation (4.2) *if and only if there exists a subspace $W' \subset W_\tau$ of codimension n such that $\Lambda^k W' \subset W_\tau$.*

Proof. Notice first that $\Lambda^k W_\tau = W_{Q^{-k}\tau}$. We assume that n is minimal, so that all σ_j and ρ_j are nonzero perfect wedges, and that τ is of the form (4.5). Then

$$\begin{aligned}\sum_{i\in\mathbb{Z}+\frac12} &\psi_i^+\tau \otimes \psi_{-i}^- Q^{-k}\tau \\ &= \sum_{\ell=0}^{\infty} (-)^\ell \Lambda^k w_{m-\ell-\frac12} \wedge \tau \otimes \Lambda^k w_{m_{\frac12}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac12} \wedge \Lambda^k w_{m-\ell-\frac32} \wedge \cdots \\ &= \sum_{j=1}^{n} u_j \wedge \tau \otimes \sigma_j,\end{aligned}$$

where $\rho_j = u_j \wedge \tau$. Since all vectors $\Lambda^k w_{m_{\frac12}} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac12} \wedge \Lambda^k w_{m-\ell-\frac32} \wedge \cdots$ are linearly independent, we deduce that

$$\Lambda^k w_{m-\ell-\frac12} \wedge u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau = 0,$$

for all $\ell = 0, 1, 2, \ldots$. Since we have assumed that n is minimal, also all u_j's are linearly independent and moreover $u_1 \wedge u_2 \wedge \cdots \wedge u_n \wedge \tau \ne 0$, hence

$$\Lambda^k w_{m-\ell-\frac12} \in \langle u_1, u_2, \ldots, u_n, w_{m-\frac12}, w_{m-\frac32}, \ldots \rangle,$$

so there exists a subspace $W' \subset W_\tau$ of codimension n such that $\Lambda^k W' \subset W_\tau$.

For the converse, choose a basis $w_{m-n-\frac12}, w_{m-n-\frac32}, \ldots$ of W' and extend it to a basis $w_{m-\frac12}, w_{m-\frac32}, \ldots, w_{m-n+\frac12} w_{m-n-\frac12}, w_{m-n-\frac32}, \ldots$ of W_τ, then

$$\begin{aligned}\sum_{i\in\mathbb{Z}+\frac12} &\psi_i^+\tau \otimes \psi_{-i}^- Q^{-k}\tau \\ &= \sum_{\ell=0}^{\infty} (-)^\ell \Lambda^k w_{m-\ell-\frac12} \wedge \tau \otimes \Lambda^k w_{m-\frac12} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac12} \wedge \Lambda^k w_{m-\ell-\frac32} \wedge \cdots \\ &= \sum_{\ell=0}^{n-1} (-)^\ell \Lambda^k w_{m-\ell-\frac12} \wedge \tau \otimes \Lambda^k w_{m-\frac12} \wedge \cdots \wedge \Lambda^k w_{m-\ell+\frac12} \wedge \Lambda^k w_{m-\ell-\frac32} \wedge \cdots .\end{aligned}$$

So choose

$$\begin{aligned}\rho_j &= \Lambda^k w_{m-j+\frac12} \wedge \tau, \\ \sigma_j &= \Lambda^k w_{m-\frac12} \wedge \cdots \wedge \Lambda^k w_{m-j+\frac32} \wedge \Lambda^k w_{m-j-\frac12} \wedge \cdots ,\end{aligned}$$

then $W_\tau,\ \Lambda^k W_\tau,\ W_{\sigma_j}$ and W_{ρ_j} clearly satisfy the equations (4.9). $\square$

From this proposition we deduce the main Theorem of this paper

Theorem 4.4. *The pseudo-differential operator*

$$L = \partial + u_1\partial^{-1} + u_2\partial^{-2} + \cdots ,$$

satisfies the n–vector k–constrained KP hierarchy if and only if the corresponding point $W \in Gr_m$ has a subspace W' of codimension n such that $t^k W' \subset W$.

As an easy consquence we obtain

Corollary 4.5. *Let τ be a polynomial τ–function of the n–vector k–constrained KP hierarchy, then $\frac{\partial\tau}{\partial t_k} = \sum_{\ell=1}^{n} \tau_\ell$ where every τ_ℓ satisfies the KP hierarchy, i.e. equation* (4.1).

Proof. The proof follows immediately by taking the same basis for W_τ as in the converse part of the proof of Proposition 4.3. □

If $n = 1$, one can proof [V] that every polynomial τ–function τ, for which $\frac{\partial\tau}{\partial t_k}$ is again τ–function, is a solution of the k–constrained KP hierarchy.

Notice that we have constructed a natural filtration on the space Gr_m, which is determined by the n–vector k–constrained KP hierarchy for $n = 0, 1, 2, \ldots$. Let

$$\begin{aligned} Gr_m^{(n,k)} = \{W \in Gr_m | &\text{there exists a subspace } W' \subset W \\ &\text{of codimension } n \text{ such that } t^k W' \subset W\}, \end{aligned} \tag{4.10}$$

then

$$Gr_m^{(0,k)} \subset Gr_m^{(1,k)} \subset \cdots \subset Gr_m^{(n,k)} \subset Gr_m^{(n+1,k)} \subset \cdots . \tag{4.11}$$

It is obvious that every point $W \in Gr_m$ (in this polynomial case) is contained in $Gr_m^{(n,k)}$ for $n >> 0$, in other words

$$Gr_m = \bigcup_{n\in\mathbb{Z}_+} Gr_m^{(n,k)}. \tag{4.12}$$

So for every τ–function of the KP hierarchy there exists a non–negative integer n such that for all $m \geq n$, τ is also a τ–function of the m–vector k–constrained KP hierarchy. In other words, for every L, corresponding to a polynomial τ–function, one can find a non–negative integer n such that L satisfies (1.3).

§5. Polynomial solutions of the n–vector k–constrained KP hierarchy..

We will now state an immediate consequence of the boson–fermion correspondence, viz., we calculate the immage under σ of a perfect wedge of the form (2.9). One finds the folowing result

Proposition 5.1. *Let S_i be the elementary Schur functions, defined by* $\exp\sum_{i=1}^\infty t_iz^i = \sum_{i\in\mathbb{Z}}^\infty S_i(t)z^i$ *($S_i=0$ for $i<0$) and let $\tau_m\in\mathcal{O}_m$ be of the form (2.9), i.e.,*

$$\tau_m = A_{m-\frac12}\wedge A_{m-\frac32}\wedge A_{m-\frac52}\wedge\cdots$$

with $A_j=\sum_{i\in\mathbb{Z}+\frac12}A_{ij}v_i$ and $A_{-k}=v_{-k}$ for all $k>N>>0$. Set $A=(A_{ij})_{i\in\mathbb{Z}+\frac12,m>j\in\mathbb{Z}+\frac12}$ and let $\Lambda=\sum_{i\in\mathbb{Z}=\frac12}E_{i,i+1}\in\overline{gl_\infty}$. Then

$$\sigma(\tau_m) = det\left(\sum_{i,j=-n+\frac12}^{m-\frac12}\left(\sum_{\ell=-N+\frac12}^{\infty}S_{\ell-i}A_{\ell j}\right)E_{ij}\right)q^m. \tag{5.1}$$

Proof. The proof of this proposition is the same as the proof of Theorem 6.1 of [KR]. One computes

$$\sigma(\exp\left(\sum_{i=1}^\infty t_i\Lambda^i\right)\tau_m)$$

and takes the coefficient of q^m. One thus obtains (see also [DJKM1,M]):

$$\sigma(\tau_m) = \det\left(\left(\exp\left(\sum_{i=1}^\infty t_i\Lambda^i\right)A\right)_{<m}\right)q^m, \tag{5.2}$$

where $B_{<m}$ denotes the submatrix of B where one only takes the rows $j\in\mathbb{Z}+\frac12$ with $j<m$. Notice that $\sum_i t_i\Lambda^i\in\overline{gl_\infty}$ and $\exp(\sum_i t_i\Lambda^i)\in\overline{GL_\infty}$. Here we calculate the determinant of an infinite matrix, however there is no problem, since the matrix is of the form $(B_{ij})_{m>i,j\in\mathbb{Z}+\frac12}$ with all but a finite number of $B_{ij}-\delta_{ij}$ with $i\ge j$ are zero.

It is clear that one can subtract $\sum_{i<-N}A_{ij}v_i$ from every A_j, with $j>-N$, in τ_m, this will not change τ_m. Then the new A is of the form

$$A=\sum_{-N<i,-N<j<m}A_{ij}E_{ij}+\sum_{i<-N}E_{ii},$$

it is then straightforward, using the elementary Schur functions, to calculate the right-hand-side of (5.2). One finds formula (5.1). □

We will use this proposition to obtain all polynomial solutions of the n–vector k–constrained KP hierarchy. Notice that our approach is different from the one in [ZC]. Instead of taking τ_m of the form (2.9), we may choose another basis of W_{τ_m} and construct the corresponding perfect wedge, it is clear that this wil be a multiple of τ_m. We can choose this basis in such a way

$$W_{\tau_m}=\langle A_{m-\frac12},A_{m-\frac32},A_{m-\frac52},\dots,A_{-N+\frac12},v_{-N-\frac12},v_{-N-\frac32},\dots\rangle,$$

such that $A_j=\sum_{i=-N+\frac12}^{\infty} A_{ij}v_i$ and that, except for at most n vectors A_j, all A_j satisfy the following condition

$$\Lambda^k A_j \begin{cases} = A_\ell \text{ for some } -N+\frac12 \le \ell \le m-\frac12, \text{ or} \\ \in \langle v_{-N-\frac12}, v_{-N-\frac32}, \dots\rangle. \end{cases}$$

Of course every A_j is bounded, i.e., there exists an integer M such that all $A_j=\sum_{i=-N+\frac12}^{M-\frac12} A_{ij}v_i$. Now making a shift in the index and permuting the columns we obtain the following result:

Proposition 5.2. *Let $M,N\in\mathbb{Z}$ such that $M>N>0$ and let e_j, $1\le j\le M$ be an orthonormal basis of $\mathbb{C}^M$. Let R be the $M\times M$–matrix $R=\sum_{i=1}^{M-k} E_{i,i+k}$ and let $A=(A_{ij})_{1\le i\le M,1\le j\le N}$ be an $M\times N$–matrix of rank N. Denote by $A_j=\sum_{i=1}^M A_{ij}e_i$. If all A_j satisfy the condition that $RA_j\ne A_i$ for all $1\le i<j$ and if all A_j, except for at most n, satisfy the condition that*

$$RA_j = \begin{cases} A_{j+1} & \textit{or} \\ 0, \end{cases}$$

then

$$\tau = det\left(\sum_{i,j=1}^N \left(\sum_{\ell=1}^M S_{\ell-i}A_{\ell j}\right)E_{ij}\right) \tag{5.3}$$

is a τ–function of the n–vector k–constrained KP hierarchy. All polynomial solutions can be obtained in this way.

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References

[**AFGZ**] H. Aratyn, L. Ferreira, J.F. Gomes, A.H. Zimerman, *Constrained KP models as integrable matrix hierarchies*, hep-th 9509096.

[**C**] Yi Cheng, *Modifying the KP, the n^{th} constrained KP hierarchies and their hamiltonian structures*, Commun. Math. Phys. **171** (1995), 661–682.

[**CSZ**] Yi Cheng, Walter Strampp, Bin Zhang, *Constraints of the KP hierarchy and multilinear forms*, Commun. Math. Phys. **168** (1995), 117–135.

[**CZ**] Yi Cheng, You-jin Zhang, *Bilinear equations for the constrained KP hierarchy*, Inverse Problems **10** (1994), L11–L17.

[**DJKM1**] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Operator approach to the Kadomtsev-Petviashvili equation. Transformation groups for soliton equations. III*, J. Phys. Soc. Japan **50** (1981), 3806–3812.

[**DJKM2**] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations. Euclidean Lie algebras and reduction of the KP hierarchy*, Publ. Res. Inst. Math. Sci. **18** (1982), 1077–1110.

[**DJKM3**] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Transformation groups for soliton equations*, in: Nonlinear integrable systems—classical theory and quantum theory eds M. Jimbo and T. Miwa, World Scientific, 1983, 39–120.

[**D**] L.A. Dickey, *On the constrained KP (preprint).*

[**DS**] L. Dickey, W. Strampp, *On new identities for KP Baker functions and their application to constrained hierarchies*, (preprint).

[**FK**] I.B. Frenkel and V.G. Kac, *Basic representations of affine Lie algebras and dual resonance models*, Invent. Math. **62** (1980), 23–66.

[**JM**] M. Jimbo and T. Miwa, *Solitons and infinite dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. **19** (1983), 943–1001.

[**KP1**] V.G. Kac and D.H. Peterson, *Spin and wedge representations of infinite–dimensional Lie algebras and groups*, Proc. Natl. Acad. Sci. USA **78** (1981), 3308-3312.

[**KP2**] V.G. Kac and D.H. Peterson, *Lectures on the infinite wedge representation and the MKP hierarchy*, Sem. Math. Sup., vol. 102, Presses Univ. Montreal, Montreal, 1986, pp. 141–184.

[**KR**] V.G. Kac and A.K. Raina, *Bombay lectures on highest weight representations of infinite-dimensional Lie algebras*, Advanced Ser. in Math. Phys., vol. 2, World Scientific, 1987.

[**KV**] V.G. Kac and J.W. van de Leur, *The n–component KP hierarchy and representation theory.*, in Important Developments in Soliton Theory, eds. A.S. Fokas and V.E. Zakharov. Springer Series in Nonlinear Dynamics, 1993, 302–343.

[**LW**] Ignace Loris, Ralph Willox, *Bilinear form and solutions of the k–constrained Kadomtsev–Petviashvili hierarchy*, (preprint).

[**M**] M. Mulase, *Algebraic theory of the KP equations*, in Perspectives in Mathematical Physics, eds. R Penner and S.-T. Yau, International Press Company (1994), 157-223.

[**OS**] W. Oevel, W. Strampp, *Constrained KP hierarchies and bi– hamiltonian structures*, Commun. Math. Phys. **157** (1993), 51–81.

[**S**] M. Sato, *Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds*, Res. Inst. Math. Sci. Kokyuroku **439** (1981), 30–46.

[**SS**] J. Sidorenko and W. Strampp, *Multicomponent integrable reductions in the Kadomtsev–Petviashvilli hierarchy*, J. Math Phys. **34** (1993), 1429–1446.

[**SW**] G. Segal and G. Wilson, *Loop groups and equations of KdV type*, Inst. Hautes Etudes Sci. Publ. Math. **63** (1985), 1–64.

[**TV**] F. ten Kroode and J. van de Leur, *Bosonic and fermionic realizations of the affine algebra* $\hat{gl}_n$, Comm. Math. Phys. **137** (1991), 67–107.

[**V**] J. van de Leur, *A geometrical interpretation of the constrained KP hierarchy*, (preprint).

[**ZC**] You–jin Zhang and Yi Cheng, *Solutions for the vector k–constrained KP hierarchy*, J. Math Phys. **35** (1994), 5869–5884.